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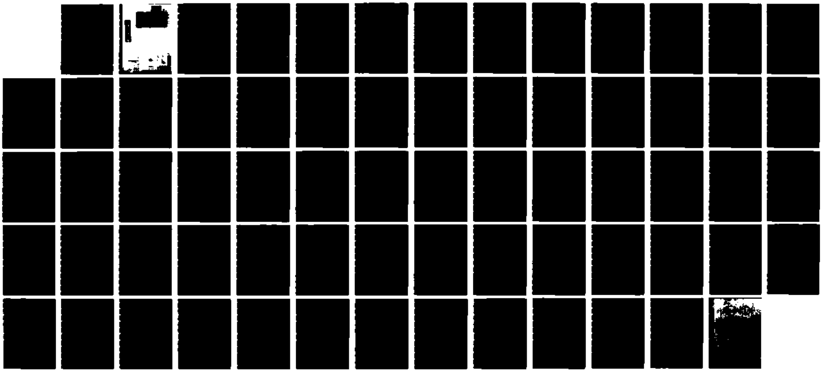

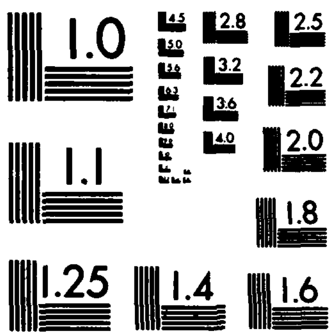
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SIGNAL DETECTION FOR SAME-SHAPE FAMILIES OF NON-HOMOGENEOUS POISSON PROCESSES

Roger Lui, C. B. Bell, Lloyd Gavin
Joseph Moser, and Edward Pugh

of Mathematical Sciences

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20. ABSTRACT (Continue on reverse side if necessary and identify by block number):

Non-Gaussian signal detection problems are treated in the context of same-shape families of non-homogeneous Poisson processes. The ratio of mean functions of two processes in such a family is a constant. Five different signal detection problems and five different sampling protocols are considered. Optimal procedures based on minimal sufficient statistics are developed.

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SIGNAL DETECTION FOR SAME-SHAPE FAMILIES

OF NON-HOMOGENEOUS POISSON PROCESSES

Roger Lui*, C. B. Bell, Lloyd Gavin**, Joseph Moser, and Edward Pugh

0. Introduction and Summary

Among the non-Gaussian signal detection models, the models associated with non-homogeneous Poisson processes (NHPP's) are perhaps the most applicable and most tractable.

This paper concerns same-shape *families of* NHPP's, that is, families for which the ratio of any two mean functions is a constant. One such family is, of course, the family of homogeneous Poisson processes (HPP's). Another such family is the set of all NHPP's with mean functions of the form $\mu(t) = \beta \ln (1 + 3t)$, where $\beta > 0$. In general, one is concerned with $\Omega(\mu_o)$, which contains all NHPP laws with mean functions of the form $\mu(t) = \beta\, \mu_o(t)$, where $\beta > 0$.

For each such family, five different signal detection problems are considered. In each problem the pure noise (PN) situation is specified, and any data which indicates that something other than the specified situation has occurred, leads to the conclusion that some signal is present, i.e., one has a noise-plus-signal (N+S) situation.

These five signal detection problems are as follows.

Problem A PN: $\mathcal{L} \epsilon\, \Omega(\mu_o)$

(Goodness-of-fit with nuisance parameter or class-fit)

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<u>Problem B</u> PN: $\mathcal{L} \varepsilon \Omega(\mu_o)$ and $\mu(.) = \beta_o \mu_o(\cdot)$

(Goodness-of-fit)

<u>Problem C</u> PN: $\mathcal{L}_1 = \mathcal{L}_2 \; [\varepsilon \; \Omega(\mu_o)]$

(2-sample problem)

<u>Problem D</u> PN: $\mathcal{L} \varepsilon \Omega(\mu_o)$ and $\mu_1(\cdot) = \alpha_o \mu_2(\cdot)$

(modified 2-sample problem)

<u>Problem E</u> $\mathcal{L} \varepsilon \Omega(\mu_o)$ and $\mathcal{L}_1 = \mathcal{L}_2 = \ldots = \mathcal{L}_c$

(c-sample problem)

For each of these (non-Gaussian) signal detection problems, one considers five different data collection schemes or sampling plans. At least one detection rule is developed for every one of these 25 sampling-plan-problem combinations.

Section 1 introduces the notation, distributions, etc. Section 2 treats the sampling plans and their associated likelihood functions. Sections 3, 4, 5, 6, and 7 treat detection problems A, B, C, D and E, respectively. Finally, in the appendix, numerical illustrations of each technique are given.

1. Statistical Preliminaries

Most of the distributions and concepts employed in this paper are fairly standard. However, it is worthwhile to give the more common notation and abbreviations one will encounter.

1.1 Notation

(1) MSS = minimal sufficient statistic

(2) MLE = maximum likelihood estimate

(3) MVUE = minimum variance unbiased estimate

(4) LHR = likelihood ratio

(5) GOF = goodness-of-fit

(6) HPP = homogeneous Poisson Process

(7) NHPP = non-homogeneous Poisson Process

(8) PN = pure noise

(9) N+S = noise plus signal

(10) K-S = Kolmogorov-Smirnov

(11) N-B = Neyman-Barton

(12) $\Omega(\mu_0)$ = the family of laws of NHPP's with mean functions of the form $\mu(\cdot) = \beta\ \mu_0(\cdot)$

1.2 Methodology. The paper concerns NHPP's and inference relative to their parameter values. The situation is that when there is PN (pure noise), the parameter is in a certain range of values and when there is signal present, i.e., the N+S case, the parameter is in some other range of values.

This being the case, one will be concerned with MSS's (minimal sufficient statistics) for the parameters, as well as MLE's (maximum likelihood estimates) and MVUE's (minimum variance unbiased estimate).

The detection procedures will be based on reducing the detection problems to GOF (goodness-of-fit), 2-sample, and c-sample statistical problems. The GOF problems will be treated with K-S (Kolmogorov-Smirnov) and N-B (Neyman-Barton) type statistics. (See Section 1.3 below) For the 2-sample and c-sample problems, the detection procedures will be LHR (likelihood ratio) procedures, or conditional procedures.

For each of the detection problems, the number of possible procedures is quite large. In this paper, only one or two procedures is given for each problem. These procedures were chosen on the basis of optimality and simplicity. Some of the references in the bibliography contain alternate statistical techniques which can be adapted to signal detection.

1.3 Statistical Distributions. The signal detection models considered in this paper involve only NHPP's. However, for each of the detection problems five different sampling plans are considered--

Plan A (Type I Censoring);

Plan B (Type II Censoring);

Plan C (Regular Sampling);

Plan D (Equal-Distance Sampling); and

Plan E (Same-Shape Sampling).

A variety of standard statistical distributions come into play. They are as follows:

(1) $Exp(\lambda)$ = Exponential distribution with mean λ^{-1}

(2) $P_o(\lambda)$ = Poisson distribution with mean λ

(3) χ^2_m = Chi-square distribution with "m" degrees of freedom

(4) $Ge(p)$ = Geometric distribution with parameter p

(5) $U(0,\theta)$ = Uniform distribution on $(0,\theta)$

(6) $\Gamma(n,\lambda)$ = Gamma distribution with parameters n and λ

(7) $B(m,p)$ = Binomial distribution with parameters m and p

(8) $F(m,k)$ = Fisher's F-distribution with m and k degrees of freedom, respectively

(9) $N\text{-}B(k,p)$ = Negative Binomial distribution with parameters k and p

(10) $\varepsilon(\cdot)$ = the one-point distribution with all probability at 0, i.e. $\varepsilon(u) = 1$ for $u \geq 0$; $= 0$ for $u < 0$.

Besides the distributions above, one will be concerned with several distributions based on random samples, $X_1, \ldots, X_m$ i.i.d. $G(\cdot)$, with order statistics $X(1), \ldots, X(m)$.

(a) G-O-S(m) = the distribution of the vector $[X(1), \ldots, X(m)]$

(b) U-O-S(m) = the distribution above when $G = U(0,1)$

(c) EDF = empirical distribution function, $F_n(z) = m^{-1} \sum_1^m \varepsilon(z - X_j)$ and $\varepsilon(\cdot)$ is as in (10) above. This is a mixture of one-point distributions.

(d) K-S(m) = the distribution of the Kolmogorov-Smirnov statistic $D(G,m) = \sup_z \mid F_m(z) - G(z) \mid$.

(e) $\mathcal{N}\text{-}\mathcal{B}(k)$ = the (exact) distribution of the Neyman-Barton statistic $(\underset{\sim}{V} - \underset{\sim}{\mu})\, \tilde{\Sigma}^{-1} (\underset{\sim}{V} - \underset{\sim}{\mu})'$, where $\underset{\sim}{\mu} = [\frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{k+1}]$, $\underset{\sim}{V} = [\bar{V}^1, \bar{V}^2, \ldots, \bar{V}^k]$ with $\bar{V}^r = k^{-1} \sum_1^k X^r$, and $\tilde{\Sigma} = (\sigma_{rs})$, with $\sigma_{rs} = rs\, [(r+1)(s+1)(r+s+1)]^{-1}$.

[This distribution is asymptotically χ_k^2.]

For each of these distributions one writes, e.g., $X \sim G$, to indicate that G is the distribution of X; and $[Y_1, \ldots, Y_m] \sim$ G-O-S(m), when the vector has the distribution in (a) above.

One now introduces the five sampling plans for dealing with the NHPP's. Several of these plans are "natural." For example, it is often necessary (for a variety of reasons) to stop sampling at some fixed time T^*, or to stop after the occurrence of some fixed number of events.

Other times, it is desirable to sample in such a way that certain distributional and computational difficulties are surmounted. Such is the case with equal-distance and same-shape sampling.

All of these plans, their associated likelihood functions, MSS's and MLE's are treated in the next section.

2. Sampling Plans, Likelihood Functions MLE and MSS's

Five different sampling plans will be considered.

2.1 Sampling Plan A: Type I Censoring. The stopping rule here is: Stop a time T^*. The data is then,

$$\underset{\sim}{Z} = [k, W_1, \dots, W_k]$$

where $k = N(T^*)$. It is well known that

Theorem 2.1.1 Conditionally, given $N(T^*) = k$,

(a) $\underset{\sim}{W} = [W_1, \dots, W_k] \sim G_{\mu_o}$-O-S(k),

where $G_{\mu(t)} = \dfrac{\mu(t)}{\mu(T^*)}$, for $0 \leq t \leq T^*$, and

(b) $\left[\dfrac{\mu_o(W_1)}{\mu_o(T^*)}, \dots, \dfrac{\mu_o(W_k)}{\mu_o(T^*)}\right] \sim$ U-O-S(k).

(c) Further, the likelihood function is

$$L(\underset{\sim}{z}) = L(\underset{\sim}{w} \mid N(T^*) = k) \quad P\{ N(T^*) = k\} = \beta^k \left[\prod_1^k \mu_o'(w_j)\right] \exp\{-\beta\,\mu_o(T^*)\}.$$

From the likelihood function, one derives the following.

Theorem 2.1.2 (a) The MSS for β is $N(T^*) \sim P_o(\beta\mu_o(T^*))$.

(b) The MLE of β is $\hat{\beta} = \dfrac{N(T^*)}{\mu_o(T^*)}$ and

(c) $E(\hat{\beta}) = \beta$ and $V(\hat{\beta}) = \dfrac{\beta}{\mu_o(T^*)}$.

The signal detection methodology for this sampling plan will employ the previous two theorems.

2.2 Sampling Plan B: Type II Censoring

The stopping rule here is: Stop at W_N, the Nth waiting time. The data is, then, $\underset{\sim}{W} = [W_1, \dots, W_N]$.

The principal distribution theorem is analagous to Theorem 2.1.1 above.

Theorem 2.2.1. (a) Conditionally, given $W_n = w^*$,

$\underset{\sim}{W}^* = [W_1, \ldots, W_{N-1}] \sim G_{\mu_o}\text{-O-S}(N-1)$, and

(b) $[\frac{\mu_o(W_1)}{\mu_o(W_n)}, \ldots, \frac{\mu_o(W_{n-1})}{\mu_o(W_n)}] \sim \text{U-O-S}(N-1)$.

(c) Further, the likelihood function is

$$L(\underset{\sim}{w}) = L(w_1, \ldots, w_{N-1} \mid W_N = w^*)\, f_{W_N}(w^*)$$

$$= \beta^N [\prod_1^N \mu_o'(W_j)] \exp \{-\beta\mu_o(w^*)\} .$$

Employing the likelihood function, one obtains

Theorem 2.2.2 (a) The MSS of β is $\mu_o(W_n) \sim \Gamma(N,\beta)$.

Further, $2\beta\mu_o(W_N) \sim \chi^2_{2N}$.

(b) The MLE of β is $\hat{\beta} = \frac{N}{\mu_o(W_N)}$ with $E(\hat{\beta}) = \frac{N\beta}{N-1}$

and $V(\hat{\beta}) = \frac{N^2\beta^2}{(N-1)^2(N-2)}$

(c) The UMVUE of β is $\beta^* = \frac{N-1}{\mu_o(W_N)} = (\frac{N-1}{N})\hat{\beta}$, with

$E(\beta^*) = \beta$ and $V(\beta^*) = \frac{\beta^2}{N-2}$

The signal detection procedures for this sampling plan will primarily be based on the two preceding theorems.

2.3 Sampling Plan C: Regular Sampling. Here, one observes the process at times $\Delta, 2\Delta, \ldots, k\Delta$, and the data is $\underset{\sim}{Y} = [Y_1, \ldots, Y_k]$ where $Y_1 = N(\Delta)$, $Y_2 = N(2\Delta) - N(\Delta)$, ... $Y_k = N(k\Delta) - N([k-1]\Delta)$.

The basic distribution results are as follows:

Theorem 2.3.1 (a) $Y_1, \ldots, Y_k$ are independent.

(b) $Y_j \sim P_o(b_j\beta)$, where $b_j = \mu_o(j\Delta) - \mu_o([j-1]\Delta)$, for $j = 1, 2, \ldots, k$.

(c) The likelihood function is

$$L(y) = \beta^{kN(k\Delta)} \left[\prod_1^k b_j^{y_j} \right] \left[\prod_1^k y_j! \right]^{-1} \exp \{- \beta\mu_o(k\Delta)\}$$

One uses the likelihood function to establish

Theorem 2.3.2 (a) The MSS for β is $N(k\Delta) \sim P_o(\beta\mu_o(k\Delta))$

(b) The MLE of β is $\hat{\beta} = \dfrac{N(k\Delta)}{\mu_o(k\Delta)}$

(c) $E(\hat{\beta}) = \beta$ and $V(\hat{\beta}) = \dfrac{\beta}{\mu_o(k\Delta)}$

As is usual, these last two theorems form the basis for the signal detection procedures for regular sampling.

2.4 Sampling Plan D: Equal-Distance Sampling. In this sampling plan one observes $N(t_1)$, $N(t_2)$, ..., $N(t_k)$ where $\mu_o(t_r) = r\ \mu_o(t_1)$. The data here is $\underset{\sim}{Y} = [Y_1, \ldots, Y_k]$, where $Y_r = N(t_r) - N(t_{r-1})$ for $r = 1, 2, \ldots, k$ and $t_o = 0$.

The pertinent distribution result is

Theorem 2.4.1 (a) $Y_1, \ldots, Y_k$ are i.i.d. $P_o(\beta\mu_o(t_1))$

(b) The likelihood function is

$$L(\underset{\sim}{y}) = [\beta\mu_o(t_1)]^{N(t_k)} \left[\prod_1^k y_j! \right]^{-1} \exp \{- k\beta\mu_o(t_1)\}.$$

Immediate consequences are as follows.

Theorem 2.4.2 (a) The MSS for β is $N(t_k) \sim P_o(\beta\mu_o(t_k))$.

(b) The MLE of β is $\hat{\beta} = \dfrac{N(t_k)}{\mu_o(t_k)} = \dfrac{N(t_k)}{k\mu_o(t_1)}$

(c) $E(\hat{\beta}) = \beta$ and $V(\hat{\beta}) = \dfrac{\beta}{\mu_o(t_k)}$

These last two theorems will be utilized in constructing the detection procedures for the equal-distance sampling plan.

The final sampling plan is the same-shape sampling plan adapted from Basawa and Rao (1980), who were concerned with homogeneous Poisson Processes (HPP's).

2.5 Sampling Plan E: Same-Shape Sampling. Let $\{N(t)\}$, the NHPP of interest, have mean function $\beta\mu_o(\cdot)$ and waiting times $\{W_r\}$, and let $\{N^*(t)\}$ be an NHPP independent of $\{N(t)\}$ and having mean function $\mu_o(\cdot)$ and waiting times $\{W_r^*\}$.

One observes $\underset{\sim}{N}^* = [N(W_1^*), \ldots, N(W_k^*)]$ and computes $\underset{\sim}{Y} = [Y_1, \ldots, Y_k]$, $Y_1 = N(W_1^*)$, $Y_2 = N(W_2^*) - N(W_1^*)$, $\ldots, Y_k = N(W_k^*) - N(W_{k-1}^*)$.

It can be readily proved that

Theorem 2.5.1 (a) $Y_1, \ldots, Y_k$ are i.i.d. Ge(p) with $p = [1 + \beta]^{-1}$.

(b) $\sum_1^k Y_r = N(W_k^*) \sim$ N-B(k,p).

(c) The likelihood function

$$L(y) = p^k q^{N(W_k^*)} = \beta^{N(W_k^*)} [1 + \beta]^{-k-N(W_k^*)}$$

It follows that

Theorem 2.5.2 (a) The MSS for β is $N(W_k^*)$.

(b) The MLE of β is $\hat{\beta} = \frac{N(W_k^*)}{k} = \bar{Y}$.

(c) $E(\hat{\beta}) = \beta$ and $V(\hat{\beta}) \quad \frac{\beta(1+\beta)}{k}$

One is now in a position to develop the various signal detection procedures.

The detection rules will be designated--first, by problem; second, by sampling plan; and third, by the number. For example, "Detection Rule B.II.1" refers to the first detection rule for Detection Problem B with Sampling Plan II; and "Detection Rule C.III.2" refers to the second detection rule for Problem C and Sampling Plan III.

3. Signal Detection for Problem A: Class-fit, or Goodness-of-fit with Nuisance Parameter.

The situation here is that, if there is pure noise, the NHPP law is in $\Omega(\mu_o)$, while if signal if present, what is received is governed by a law not in $\Omega(\mu_o)$. One has

$$\underline{PN : \mathcal{L} \in \Omega(\mu_o)} \quad \text{vs} \quad \underline{N{+}S : \mathcal{L} \notin \Omega(\mu_o)} .$$

The decision procedures, of course, depend on the sampling plan.

3.1 Sampling Plan A (Type I Censoring) Here one stops at time T^*, where $N(T^*) = k$, and computes $\underset{\sim}{V} = [V_1, \ldots, V_k]$ where $V_r = \dfrac{\mu_o(W_r)}{\mu_o(T^*)}$.

Then, as in Theorem 2.1.1, $\underset{\sim}{V} \underset{PN}{\sim} \text{U-O-S}(k)$.

Hence, the class-fit problem is now a GOF problem.

One now forms, $F_k^*(z) = \frac{1}{k} \sum_1^k \varepsilon(z - V_r)$; $\overline{V^s} = \frac{1}{k} \sum_1^k V_j^s$, and

$\underset{\sim}{\overline{V}} = [\overline{V^1}, \overline{V^2}, \ldots, \overline{V^p}]$, and bases the decision rules on these entities.

Detection Rule A.I.1 Decide N + S iff $D_k = \sup_{o<z<1} | F_k^*(z) - z | > d_{(k,\alpha)}$,

where $d_{(k,\alpha)}$ is the appropriate percentile from a K-S(k) distribution.

This K-S detection rule is considered an "omnibus" rule in the sense that it is consistent against "all" alternatives.

Adaptation of the work of Neyman (1937) and Barton (1953) yield procedures recommended for certain "smooth" N + S situations.

Detection Rule A.I.2 Decide N + S iff

$B_p = k\ (\underset{\sim}{V} - \underset{\sim}{\mu})\ \tilde{\Sigma}^{-1}\ (\underset{\sim}{V} - \underset{\sim}{\mu})' > \chi_p^2(1-\alpha)$, where $\underset{\sim}{\mu} = [\frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{p+1}]$ and $\tilde{\Sigma} = (\sigma_{rs})$, with $\sigma_{rs} = \dfrac{rs}{(r+1)(s+1)(r+s+1)}$.

The detection procedures are different for different sampling plans. However, for this detection problem, they are similar for the first two sampling plans.

3.2 Sampling Plan II (Type II Censoring) The data here is $\underset{\sim}{W} = [W_1, \ldots, W_N]$, and under PN $[V_1^*, \ldots, V_{N-1}^*] \sim$ U-O-S (N-1), where $V_r^* = \frac{\mu_o(W_r)}{\mu_o(W_N)}$.

The detection rules here are

Detection Rule A.II.1 Same as Detection Rule 3.1.1 where k is replaced by N-1 and V's are replaced by V^*'s.

Detection Rule A.II.2 Same as Detection Rule 3.1.2, where V's are replaced by V^*'s.

3.3 Sampling Plan III (Regular Sampling) For this sampling plan, $Y_1, \ldots, Y_k$ are independent with $Y_j \sim P_o(b_j\beta)$ under PN (See Section 2.3).

A useful straightforward detection rule is

Detection Rule A.III.1 Decide N + S iff

$$T = \sum_{j=1}^{k} \frac{[y_j - \frac{b_j}{\bar{b}}\bar{y}]^2}{\frac{b_j}{\bar{b}}\bar{y}} =$$

$$[\mu_o(k\Delta)\, N(k\Delta)]^{-1} \sum_{j=1}^{k} b_j^{-1}\, [y_j\, \mu_o(k\Delta) - b_j\, N(k\Delta)]^2 > \chi^2_{k-1}\,(\alpha).$$

For the next sampling plan, the increments y_j are i.i.d., and, hence, the detection procedure is somewhat simpler.

3.4 Sampling Plan IV (Equal-Distance Sampling) One recalls (Section 2.4) that $Y_1, \ldots, Y_k$ are i.i.d. $P_o(\beta\mu_o(t_1))$, where $Y_r = N(t_r) - N(t_{r-1})$, $t_o = 0$, and $\mu_o(t_r) = r\mu_o(t_1)$. A decision rule of interest is then

Detection Rule A.IV.1 Decide N + S iff

$$\chi^2_{k-1}(1-\alpha) < T^* = (\bar{Y})^{-1} \sum_{j=1}^{k} (Y_j - \bar{Y})^2$$

$$= \frac{k}{N(t_k)} \sum_{r=1}^{k} [N(t_r) - N(t_{r-1}) - \frac{N(t_k)}{k}]^2$$

Each of these rules is, of course, illustrated by a numerical example in the appendix.

The final detection rule for this section is based on same-shape sampling.

3.5 Sampling Plan V (Same-Shape Sampling) For this sampling plan, under PN the Y's are i.i.d. Ge(p) where $p = [1+\beta]^{-1}$, and $Y_r = N(W_r^*) - N(W_{r-1}^*)$ with $W_o^* = 0$. (See Section 2.5)

Following usual procedures one develops

Detection Rule A.V.1 Decide N + S iff

$$\sum_{r=1}^{s+1} \frac{[n_r - k\hat{p}(C_r)]^2}{k\hat{p}(C_r)} > \chi^2_{s-1}(1-\alpha)$$

Here, $\{A_1, \ldots, A_s\}$ are integers satisfying $A_o = 0 < A_1 < A_2 < \ldots < A_s < \infty$. Then, one chooses $C_r = \{1 + A_{r-1}, 2 + A_{r-1}, \ldots, A_r\}$ for $1 \leq r \leq s$, and $C_{s+1} = \{A_s, 2 + A_s, \ldots\}$.

Let n_r = the number of Y's in C_r, and

$$\hat{p}(C_r) = \left(\frac{\bar{Y}}{1+\bar{Y}}\right)^{1+A_{r-1}} - \left(\frac{\bar{Y}}{1+\bar{Y}}\right)^{1+A_r}, \text{ for } 1 \leq r \leq s$$

$$\text{and } \hat{p}(C_{r+1}) = 1 + \left(\frac{\bar{Y}}{1+\bar{Y}}\right)^{1+A_r} - \left(\frac{\bar{Y}}{1+\bar{Y}}\right) .$$

$\{C_1, \ldots, C_{s+1}\}$ constitute a partition of the non-negative integers, and there are many such partitions. The partition which will be used in the numerical examples of the appendix is as follows:

$C_1' = \{0,5,10,15,\ldots\}$; $C_2' = \{1,6,11,16,\ldots\}$; $C_3' = \{2,7,12,17,\ldots\}$; $C_4' = \{3,8,13,18,\ldots\}$ and $C_5' = \{4,9,14,19,\ldots\}$.

Here, $\hat{p}(C_r) = \hat{p}(\hat{q})^{r-1} [1 - (\hat{q})^5]^{-1}$ for $r = 1, \ldots, 5$, where $\hat{p} = [1+\bar{Y}]^{-1}$ and $\hat{q} = 1 - \hat{p}$.

One turns now to the second detection problem.

4. Signal Detection for Problem B: Goodness-of-Fit

The PN case here entails the stochastic process law to be completely known, i.e. to be a NHPP law with mean function $\mu(\cdot)$ satisfying $\mu(t) = \beta_o \mu_o(t)$; given that $\mathcal{L} \in \Omega(\mu_o)$.

$$\underline{PN : \mu(\cdot) = \beta_o\mu_o(\cdot)} \quad \text{vs.} \quad \underline{N{+}S : \mu(\cdot) \neq \beta_o\mu_o(\cdot)}$$

The problem here is to decide whether or not $\beta = \beta_o$.

4.1 Sampling Plan I (Type I Censoring) Under PN (Section 2.1) the MSS for β is $N(T^*) \sim P_o(\beta_o\mu_o(T^*))$. The detection rule, then is

Detection Rule B.I.1 Decide N + S iff $N(T^*) \leq a_1$ or $N(T^*) \geq a_2$,

where $\sum_{r=0}^{a_1} \frac{e^{-\lambda}\lambda^r}{r!} + \sum_{r=a_2}^{\infty} \frac{e^{-\lambda}\lambda^r}{r!} = \alpha$ with $\lambda = \beta_o\mu_o(T^*)$.

In practice, one would probably wish to choose a_1 and a_2 to have approximately equal tail probabilities under PN.

Contrary to what was true for Problem A, the detection rule for Sampling Plan II here is somewhat different from that for Plan I.

4.2 Sampling Plan II (Type II Censoring) One uses the fact that under PN, $2\beta_o\mu_o(W_N) \sim \chi^2_{2N}$, to construct a reasonable detection rule. (See Section 2.2)

Detection Rule B.II.1 Decide N + S iff $2\beta_o\mu_o(W_N) < b_1$ or $> b_2$ where b_1 and b_2 are appropriate percentiles of the χ^2_{2N} - distribution.

For the third sampling plan, one makes use of an approximate chi-square distribution.

4.3 Sampling Plan III (Regular Sampling) The data in this case is $\underset{\sim}{Y} = (Y_1, \ldots, Y_k)$, where $Y_r = N(r\Delta) - N([r-1]\Delta)$, with $N(0) = 0$. Further, one recalls (Section 2.3) that under PN $Y_1, \ldots, Y_k$ are independent with $Y_r \sim P_o(\beta b_r)$, where $b_r = \mu_o(r\Delta) - \mu_o([r-1]\Delta)$.

One useful detection rule is

Detection Rule B.III.1 Decide N + S iff

$$T = \sum_{r=1}^{k} (\beta_o b_r)^{-1} [Y_r - \beta_o b_r]^2 > \chi_k^2 (1 - \alpha).$$

Another such rule (which employs the MSS for β) is

Detection Rule B.III.2 Decide N + S iff $\sum_1^k Y_r = N(k\Delta) \leq a_1$ or $\geq a_2$

(as in Detection Rule B.I.1).

The situation is somewhat different for equal-distance sampling.

4.4 Sampling Plan IV (Equal-Distance Sampling) From Section 4.4, one recalls that $\mu_o(t_r) = rt_1$; $Y_r = N(t_r) - N(t_{r-1})$ with $N(0) = 0$; and that $Y_1, \ldots, Y_k$ are i.i.d. $P_o(\beta_o\mu_o(t_1))$ under PN, and that $N(t_k)$ is the MSS for β.

Under such circumstances one constructs

Detection Rule B.IV.1 Decide N + S iff $\sum_1^k Y_r = N(t_k) \leq a_1$ or $\geq a_2$

(as in Detection Rule B.III.2).

4.5 Sampling Plan V (Same-Shape Sampling) The M-S-S for β (Section 2.5) $\sum_1^k Y_r = N(W_k^*)$, which under PN, $\sim$ N-B(k,p) for $p = [1 + \beta_o]^{-1}$.

Under these circumstances one employs the detection rule below.

<u>Detection Rule B.V.1</u> Decide N + S iff $\sum_{1}^{k} Y_r = N(W_k^*) \leq b_1$ or $\geq b_2$,

where b_1 and b_2 are appropriate percentiles of the Negative Binomial distribution.

This final detection rule as well as the other rules of this section are illustrated in the appendix.

For the next detection problem, the rules are quite different in the sense that most of them are conditional rules.

5. Signal Detection for Problem C (2-sample)

For this problem it is assumed that data from two processes with laws in $\Omega(\mu_o)$ are being received. The PN situation entails the equality of the two laws, which is equivalent to the equality of the mean functions $\beta_1\mu_o$ and $\beta_2\mu_o$.

$$\underline{\text{PN}: \mathcal{L}_1 = \mathcal{L}_2 \text{ (i.e. } \beta_1 = \beta_2)} \quad \text{vs} \quad \underline{\text{N + S}: \mathcal{L}_1 \neq \mathcal{L}_2}$$

5.1 Sampling Plan I (Type I Censoring) Let $\{N(t)\}$ and $\{M(t)\}$ be the two independent NHPP processes and $\{W_r\}$ and $\{W_r^*\}$ be their respective waiting times. Observation of both processes is in the time interval $[0,T^*]$. The data is then (following Section 2.5)

$$\underset{\sim}{Z} = [n; W_1, \ldots, W_n; m; W_1^*, \ldots, W_m^*]$$

where $n = N(T^*)$, $m = M(T^*)$.

$N(T^*)$ and $M(T^*)$ are the M-S-S's for β_1 and β_2, respectively. They are independent with $N(T^*) \sim P_o(\beta_1\mu_o(T^*))$ and $M(T^*) \sim P_o(\beta_2\mu_o(T^*))$.

Hence,

Lemma 5.1.1 Under PN, and conditionally, given $N(T^*) + M(T^*) = N$, $N(T^*) \sim B(N,\frac{1}{2})$.

The "natural" detection rule is then

Detection Rule C.I.1 Decide N + S iff $N(T^*) \leq c_1$ or $\geq c_2$, where c_1 and c_2 are the appropriate percentiles of a $B(N,\frac{1}{2})$ distribution.

This detection rule is, as usual, illustrated by a numerical example in the appendix.

For the second sampling plan, the detection rule is based on an F-distribution.

5.2 Sampling Plan II (Type II Censoring) For this plan the data is $\underset{\sim}{Z} = [W_1, \ldots, W_n; W_1^*, \ldots, W_m^*]$, i.e. one observes the first n waiting times of $\{N(t)\}$ and the first m waiting times of $\{M(t)\}$. Then, one has

Lemma 5.2.1 Under PN, $\dfrac{m\mu_o(W_n)}{n\mu_o(W^*)} \sim F(2n,2m)$.

The "natural" detection rule here is then

Detection Rules C.II.1 Decide N + S iff $\dfrac{m\mu_o(W_n)}{n\mu_o(W_m^*)} < f_1$ or $> f_2$, where f_1 and f_2 are appropriate percentiles of an $F(2n,2m)$ - distribution.

For the next two sampling plans, the detection rules are based on conditional binomial distributions as in Section 5.1 above.

5.3 Sampling Plan III (Regular Sampling) The M-S-S's for the β_1 and β_2 are $N(n\Delta_1)$ and $M(m\Delta_2)$, respectively (Section 2.3). Further, it is immediate that

Lemma 5.3.1 Conditionally, given $N(n\Delta_1) + M(m\Delta_2) = N$, $N(n\Delta_1) \sim B(N,p)$, where $p = \dfrac{\mu_o(n\Delta_1)}{\mu_o(n\Delta_1) + \mu_o(m\Delta_2)}$

[Note: If $\Delta_1 = \Delta_2$ and $m = n$, then $p = \frac{1}{2}$.]

The detection rule is then

Detection Rule C.III.1 Decide N + S iff $N(n\Delta_1) < a_1$ or $> a_2$ (where a_1 and a_2 are appropriate percentiles of the $B(N,p)$ - distribution).

An analysis rule is derived for equal-distance sampling.

5.4 Sampling Plan IV (Equal-Distance Sampling) The data here is $\underset{\sim}{Z} = [N(t_1), \ldots, N(t_n) \;\; M(t_1), \ldots, M(t_m)]$ and the M-S-S for β_1 and β_2 are $N(t_n)$ and $M(t_m)$, respectively.

The detection rule is, then,

Detection Rule C.IV.1 Decide N + S iff $N(t_n) \leq b_1$ or $\geq b_2$ (where b_1 and b_2 are appropriate percentiles of the $B(n,p)$ - distribution), with $N = N(t_n) + M(t_m)$ and $p = \dfrac{\mu_o(t_n)}{\mu_o(t_n) + \mu_o(t_m)}$.

The final detection rule for this signal detection problem is also based on a conditional distribution. However, this conditional distribution is not binomial.

5.5 Sampling Plan V (Same Shape Sampling) The two independent stochastic processes of interest are $\{N(t)\}$ and $\{M(t)\}$ with mean functions $\beta_1\mu_o(\cdot)$ and $\beta_2\mu_o(\cdot)$, respectively. One introduces two independent NHPP's $\{N^*(t)\}$ and $\{M^*(t)\}$, which are independent of the original processes, and which have waiting times $\{W_r^*\}$ and $\{V_r^*\}$, respectively.

The data here is then

$$\underset{\sim}{Z} = [N(W_1^*), \ldots, N(W_n^*); M(V_1^*), \ldots, M(V_m^*)]$$

From Section 2.5 and some straightforward derivations, one concludes

Theorem C.5.1 (i) $N(W_n^*)$ and $M(V_m^*)$ are the M-S-S's for β_1 and β_2, respectively. (ii) Under PN, $P\{\, N(W_n^*) = s \mid N(W_n^*) + M(V_m^*) = k \,\}$

$$= \binom{n+s-1}{s}\binom{m+k-s-1}{k-s}\left[\binom{n+k-1}{k}\right]^{-1} = P(s \mid n, m, k) \text{ where } N = n + m.$$

On the basis of this result, the detection rule becomes

Detection Rule C.V.1 Decide N + S iff $N(W_n^*) < c_1$ or $> c_2$, where

$$\sum_{s=0}^{c_1} P(s \mid n, m, k) + \sum_{s=c_2}^{\infty} P(s \mid n, m, k) = \alpha.$$

6. Problem D (Constant Intensity Ratio)

This detection problem is closely related to the previous problem. Essentially, for Problem C, PN entailed the equality of the intensities of two independent NHPP's. Here, PN entails that the intensity of the first process be a fixed constant multiple of the intensity of the second process.

Under such circumstances, the data vectors and the M-S-S's for the various sampling plans are the same as in Section 5; and the detection rules are closely related to those of Section 5. For that reason, these rules will be listed with a minimum of discourse.

$$\underline{PN : \beta_1 = \gamma_o\beta_2} \quad \text{vs} \quad \underline{N + S : \beta_1 \neq \gamma_o\beta_2}$$

6.1 Sampling Plan I (Type I Censoring)

Detection Rule D.I.1 (See Rule C.I.1) Decide N + S iff $N(T^*) \leq a_1$ or $\geq a_2$, where a_1 and a_2 are appropriate percentiles of the $B(N,p)$ - distribution, with $N = N(T^*) + M(T^*)$ and $p = \dfrac{\gamma_o}{1 + \gamma_o}$

6.2 Sampling Plan II (Type II Censoring)

Detection Rule D.II.1 (See Rule C.II.1) Decide N + S iff

$$\frac{m\,\gamma_o\mu_o(W_n)}{n\,\mu_o(W_m^*)} < f_1 \text{ or } > f_2,$$ where the f's are the appropriate percentiles

of the $F(2n,2m)$ - distribution.

6.3 Sampling Plan III (Regular Sampling)

Detection Rule D.III.1 (See Rule C.III.1) Decide N + S iff $N(n\Delta_1) \leq a_1$ or $\geq a_2$, where a_1 and a_2 are appropriate percentiles of the $B(N,p)$ - distribution, with $N = n + m$ and $p = \dfrac{\gamma_o \mu_o(n\Delta_1)}{\gamma_o \mu_o(n\Delta_1) + \mu_o(m\Delta_2)}$.

6.4 Sampling Plan IV (See Rule C.IV.1) (Equal-Distance Sampling)

Detection Rule D.IV.1 Decide N + S iff $N(t_n) \leq b_1$ or $\geq b_2$, where b_1 and b_2 are appropriate percentiles of a $B(N,p)$ - distribution, with $N = N(t_n) + M(t_m)$ and $p = \dfrac{\gamma_o \mu_o(t_n)}{\gamma_o \mu_o(t_n) + \mu_o(t_m)}$.

6.5 Sampling Plan V (See Rule C.V.1) (Same-Shape Sampling)

The detection rule for this sampling plan is quite different from the other rules encountered up to this point. The crucial point here is that the M-S-S's $N(W_n^*)$ and $N(V_m^*)$ are distributed $N\text{-}B(n, \dfrac{\beta_1}{1 + \beta_1})$ and $N\text{-}B(m, \dfrac{\gamma_o\beta_1}{1 + \gamma_o\beta_1})$, respectively under PN. Hence, their sum, when $\gamma_o \neq 1$, is not distributed as a Negative Binomial variable. One, hence, resorts to a likelihood ratio procedure.

For the special case m = n = k, one has

Theorem 6.5.1 (i) The likelihood function is

$$L(y_{11}, \ldots, y_{1k}; y_{21}, \ldots, y_{2k}) = (p_1 p_2)^k \, q_1^{N(W_k^*)} \, q_2^{M(V_k^*)}$$

where $p_r = \dfrac{\beta_r}{1 + \beta_r}$ and $q_r = 1 - p_r$, for r = 1,2.

(ii) The unconditional maximum of $L(\underset{\sim}{y})$ is achieved for

$$\hat{p}_1 = [1 + \frac{N(W_k^*)}{k}]^{-1} \text{ and } \hat{p}_2 = [1 + \frac{M(V_k^*)}{k}]^{-1}$$

(iii) Under PN, $p_1 = \dfrac{p_2}{\gamma_o + p_2 (1-\gamma_o)}$; and $L(\underset{\sim}{y})$ is maximized

for $\hat{\hat{p}}_2 = [1 + \hat{x}]^{-1}$ and $\hat{\hat{p}}_1 = \dfrac{\hat{\hat{p}}_2}{\gamma_o + \hat{\hat{p}}_2 (1-\gamma_o)}$

where $\hat{x} = [4 \gamma_o]^{-1} \left(\{[1 + \gamma_o - k^{-1} N(W_k^*)]^2 + 8 \gamma_o k^{-1} [N(W_k^*) + M(V_k^*)]\}^{\frac{1}{2}}\right.$

$\left. - [1 + \gamma_o - k^{-1} N(W_k^*)]\right)$

(iv) $L(\underset{\sim}{y}, \hat{p}_1, \hat{p}_2) = [1 + \frac{N(W_k^*)}{k}]^{-k-N(W_k^*)} [1 + \frac{M(V_k^*)}{k}]^{-k-M(V_k^*)}$

$[\frac{N(W_k^*)}{k}]^{N(W_k^*)} [\frac{M(V_k^*)}{k}]^{M(V_k^*)}$

(v) $L(\underset{\sim}{y}, \hat{\hat{p}}_1, \hat{\hat{p}}_2) = \{\gamma_o \hat{\hat{p}}_2 (1-\gamma_o)[(1-\gamma_o) \hat{\hat{p}}_2 + \gamma_o]^{-1}\}^{N(W_k^*)}$

$[\hat{\hat{p}}_2 (1-\gamma_o)]^{M(V_k^*)} [(1-\gamma_o) \hat{\hat{p}}_2 + \gamma_o]^{-k}$

From standard statistical derivations, it follows that

<u>Theorem 6.5.2</u> under PN, $S^* = - 2 \ln \dfrac{(\underset{\sim}{y}, \hat{\hat{p}}_1, \hat{\hat{p}}_2)}{(\underset{\sim}{y}, \hat{p}_1, \hat{p}_2)} \approx \chi_1^2$

Therefore, the detection rule is

<u>Detection Rule D.V.1</u> Decide N + S iff $S^* < \chi_1^2 (\alpha)$.

7. Signal Detection for Problem E: c-Sample

For this problem, one has data from c independent NHPP's : $\{N_1(t)\}$, $\{N_2(t)\}$, ..., $\{N_c(t)\}$. The laws of these processes, $\mathcal{L}_1, \ldots, \mathcal{L}_c$ are all in $\Omega(\mu_0)$.

The detection problem is

$$\underline{PN} : \mathcal{L}_1 = \mathcal{L}_2 = \cdots = \mathcal{L}_c \quad \text{vs} \quad \underline{N+S} : \mathcal{L}_r\text{'s are not all equal}$$

$$(\text{i.e. } \beta_1 = \beta_2 = \cdots = \beta_c)$$

For this problem, the detection rules are much more tractable if the sample "sizes" are "equal," in the appropriate sense, and this latter condition will be assumed.

The data for the various sampling plans is as given:

Data for Sampling Plan I: $\underset{\sim}{W} = [\underset{\sim}{W}_1, \underset{\sim}{W}_2, \ldots, \underset{\sim}{W}_c]$, where the waiting times are $\underset{\sim}{W}_r = (W_{r1}, \ldots, W_{rk_r})$ and $N_r(T^*) = k_r$ for $r = 1, 2, \ldots, c$, i.e. each process is observed until time T^*.

Data for Sampling Plan II: $\underset{\sim}{W} = [\underset{\sim}{W}_1, \ldots, \underset{\sim}{W}_c]$, where $\underset{\sim}{W}_r = [W_{r1}, \ldots, W_{rN}]$ for $r = 1, 2, \ldots, c$; i.e. each process is observed until the Nth waiting time.

Data for Sampling Plan III: $\underset{\sim}{N} = [\underset{\sim}{N}_1, \ldots, \underset{\sim}{N}_c]$ or $\underset{\sim}{Y} = [\underset{\sim}{Y}_1, \ldots, \underset{\sim}{Y}_c]$ where $\underset{\sim}{N}_r = [N_r(\Delta), N_r(2\Delta), \ldots, N_r(k\Delta)]$ and $\underset{\sim}{Y}_r = [Y_{r1}, \ldots, Y_{rk}]$, with $Y_{rs} = N_r(s\Delta) - N_r([s-1]\Delta)$ and $N_r(0) = 0$ for $r = 1, 2, \ldots, c$ and $s = 1, 2, \ldots, k$.

Data for <u>Sampling Plan IV</u>: $\underset{\sim}{N} = [\underset{\sim}{N}_1, \ldots, \underset{\sim}{N}_c]$ or $\underset{\sim}{Y} = [\underset{\sim}{Y}_1, \ldots, \underset{\sim}{Y}_c]$ where $\underset{\sim}{N}_r = [N_r(t_1), \ldots, N_r(t_k)]$ and $\underset{\sim}{Y}_r = [Y_{r1}, \ldots, Y_{rk}]$, with $Y_{rs} = N_r(t_s) - N_r(t_{s-1})$ and $N_r(0) = 0$ for $r = 1, 2, \ldots, c$ and $s = 1, 2, \ldots, k$.

Data for <u>Sampling Plan V</u>: $\underset{\sim}{N} = [\underset{\sim}{N}_1, \ldots, \underset{\sim}{N}_c]$ or $\underset{\sim}{Y} = [\underset{\sim}{Y}_1, \ldots, \underset{\sim}{Y}_c]$ where, $\underset{\sim}{N}_r = [N_r(W^*_{r1}), \ldots, N_r(W^*_{rk})]$, and $\underset{\sim}{Y}_r = [Y_{r1}, \ldots, Y_{rk}]$ $Y_{rs} = N(W^*_{rs}) - N(W^*_{r,s-1})$ and $N_r(0) = 0$ for $r = 1, \ldots, c$ and $s = 1, 2, \ldots, k$. Here, $\{W^*_{1s}\}, \ldots, \{W^*_{cs}\}$ are waiting times of independent NHPP's each with mean function $\mu_o(\cdot)$ and independent of $N_1(t), \ldots, N_c(t)$.

Based on data of the type above, one finds several detection rules to be related.

<u>Detection Rules D.I.1, D.III.1 and D.IV.1</u> Decide N + S iff

$$S^* = \sum_1^c [N''_r - \bar{\bar{N}}'']^2 [\bar{\bar{N}}]^{-1} > \chi^2_{c-1}(1-\alpha).$$

For Sampling Plan I, III and IV, $N''_r = N(T^*)$, $N(k\Delta)$ and $N(t_k)$, respectively; and $\bar{\bar{N}} = c^{-1} \sum_1^c N''_r$ in each case.

For the Sampling Plan II, one employs Hartley's (1950) procedure for homoscedasticity, which is applicable here since $2\beta_r\mu_o(W_{rN}) \sim \chi^2_{2N}$

<u>Detection Rule D.II.1</u> Decide N + S iff

$[\max_r \{\mu_o(W_{rk})\}]\, [\min_r \{\mu_o(W_{rk})\}]^{-1} > h''$, the appropriate percentile of Hartley's distribution.

The final detection rule is related to determining whether or not c independent negative binomial random variables have distributions with the same probability p (of success).

Based on the likelihood-ratio criterion, one derives

Detection Rule D.V.1 Decide N + S iff $L^* > \chi^2_{c-1}(1-\alpha)$, where

$$L^* = 2ck\ \ln(k+\bar{z}) - 2c\bar{z}\ \ln\left(\frac{\bar{z}}{k+\bar{z}}\right) + 2\sum_{r=1}^{c} z_r\ \ln\left(\frac{z_r}{k+z_r}\right)$$

$z_r = N_r(W^*_{rk})$; and $\bar{z} = c^{-1}\sum_{1}^{c} z_r$.

One other detection rule is of some interest here.

Meelis (1974) considers a situation in which the N + S case involves the $\{\beta_r\}$ constituting a random sample from an appropriate continuous distribution. Based on his work, one arrives at

Detection Rule D.V.2 Decide N + S iff $\sum_{r=1}^{c} [N_r(W^*_{rk})]^2 > d^*$, where $m = \sum_{1}^{c} N_r(W^*_{rk})$ and $d^* = d^*(\alpha, c, m)$ (the appropriate percentile of Meelis' distribution).

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APPENDIX - Numerical Examples

[The calculations of the appendix were done by A. Mason.]

This appendix makes use of seven data sets. Those designated Data Set 1, 2, 3, 4 and 5 are simulated sets of data from NHPP's with the indicated mean functions. The last two data sets are records of seizures for two different epilepsy patients.

The examples are grouped by problem (designated by letters A, B, C, D and E) and by sampling plan (Roman numerals I, II, III, IV and V).

For example, "Case D.III." refers to Detection Problem D and Sampling Plan III.

	Data Set 1 $\mu(t)=2.5t^2$	Data Set 2 $\mu(t)=t^2$	Data Set 3 $\mu(t)=4t$	Data Set 4 $\mu(t)=0.5t^2$	Data Set 5 $\mu(t)=0.25t^2$
1	0.6621	1.5370	0.2640	1.6407	1.8609
2	0.8156	1.8415	0.3535	3.1687	4.2283
3	1.1033	2.2670	0.8270	3.1843	4.9514
4	1.1425	2.4329	1.4313	3.7095	6.3765
5	1.3875	2.6718	1.7190	3.7499	7.4145
6	1.4569	2.9192	2.3334	3.8316	7.6608
7	1.6535	2.9655	2.7005	4.4190	8.8394
8	1.6554	3.3051	2.9981	4.7912	10.0257
9	1.7187	3.3359	3.0349	4.8704	10.7584
10	1.8768	3.6561	3.4578	5.1042	12.6299
11	2.1242	3.6623	3.8193	5.1537	12.8330
12	2.1813	3.8465	4.1066	5.3270	12.9096
13	2.2706	3.8716	4.2841	5.6595	14.0877
14	2.3851	3.9012	4.2980	5.8857	14.9650
15	2.4161	4.4260	4.3277	5.8861	15.1432
16	2.5016	4.4830	4.4119	6.4762	15.5248
17	2.5161	4.6552	4.9939	7.0334	15.9582
18	2.5183	4.7408	5.7613	7.0720	16.2997
19	2.5231	4.8441	6.1637	7.2839	17.2403
20	2.5745	4.8968	6.4948	7.3309	17.3774
21	2.7927	4.9904	6.6561	7.6767	17.6973
22	2.9971	5.0636	7.1748	7.7333	18.6457
23	3.0902	5.0779	7.2400	8.2574	18.5428
24	3.1346	5.0933	7.3192	8.5161	18.5877
25	3.1652	5.2479	7.4199	8.3744	18.8340
26	3.1985	5.4329	7.7230	9.0870	18.8848
27	3.2867	5.4493	7.7686	9.1491	19.1638
28	3.3588	5.4559	7.9864	9.2653	19.3535
29	3.4849	5.5021	8.3356	9.7845	19.7537
30	3.5187	5.6157	9.4819	9.9437	20.0240
31	3.5476	5.6721	9.7305	10.0824	21.1768
32	3.5925	5.7419	10.1198	10.2001	21.6865
33	3.8160	5.9278	10.3810	10.4780	21.7489
34	3.8490	5.9652	10.5817	10.6291	22.2156
35	3.9115	6.0453	10.8142	10.9366	22.5020
36	3.9896	6.1611	10.9833	11.4502	23.1747
37	4.0754	6.2578	11.1056	11.6886	24.3404
38	4.1004	6.3962	11.3623	11.8080	23.7653
39	4.1185	6.3982	11.6609	11.9752	24.7081
40	4.1229	6.4602	11.7117	12.0659	25.1715
41	4.2126	6.5531	12.1708	12.2152	25.2280
42	4.2949	6.5856	12.9102	12.2362	25.3121
43	4.2966	6.5952	12.9625	12.3179	25.3591
44	4.3525	6.6012	13.6038	12.3855	25.7556
45	4.4296	6.6321	14.5538	12.4373	25.7794
46	4.4329	6.7129	15.2888	12.4553	26.0114
47	4.4472	6.7171	15.3994	12.4829	26.3572
48	4.5433	6.9320	15.4766	12.9268	26.6164
49	4.5572	7.0581	15.5336	13.0550	26.9873
50	4.5681	7.1849	15.5897	13.2268	27.1501

Epilepsy Patient 1

Start: 7:00 Stop: 19:00

SZR	TIME	DURATION (Sec)
1	7:23	3.3
2	7:35	3.5
3	7:49	4.4
4	7:50	3.5
5	9:46	4.1
6	10:04	3.3
7	10:05	2.6
8	10:22	3.0
9	10:48	5.2
10	10:50	2.8
11	10:58	3.6
12	12:46	3.4
13	14:10	2.4
14	14:57	3.2
15	16:57	3.2
16	17:18	3.4
17	17:36	3.4
18	18:07	3.4

Epilepsy Patient 2

Start: 7:50 Stop: 19:50

SZR	TIME	DURATION (Sec)
1	8:00	2.8
2	8:21	2.2
3	8:58	17.2
4	9:02	16.2
5	9:14	13.8
6	9:15	14.8
7	9:18	22.8
8	9:22	17.4
9	9:30	18.8
10	9:44	2.2
11	9:49	17.6
12	10:31	3.4
13	10:36	14.8
14	10:37	6.3
15	12:16	1.5
16	12:58	15.2
17	14:59	15.0
18	15:04	16.6
19	16:10	3.8
20	16:16	10.6
21	16:31	7.0
22	16:32	14.4
23	16:33	12.0
24	16:34	16.8
25	16:34	9.6
26	16:35	17.0
27	16:36	17.6
28	16:36	14.7
29	16:38	10.0
30	16:41	12.0
31	16:42	9.5
32	16:44	7.8
33	16:46	4.6
34	16:50	7.8
35	16:51	6.8
36	18:36	11.4
37	19:03	12.8

Problem A: Class-fit, or Goodness-of-fit with Nuisance Parameter.

$$\text{PN:} \quad L \in \Omega(\mu_0) \quad \text{vs.} \quad \text{N + S:} \quad L \notin \Omega(\mu_0)$$

where $\mu_0(t) = t^2$.

Case A.I: Observe $N(t)$ until time T^*. The data is of the form $W_1, W_2, \ldots, W_k$ where $k = N(T^*)$.

Detector Statistics: (i) Kolmogorov-Smirnov

$$D_k = \sup_{0<z<1} \left| \frac{1}{k} \sum_{j=1}^{k} \varepsilon \left[z - \frac{\mu_0(W_j)}{\mu_0(T^*)}\right] - z \right| \underset{PN}{\sim} K - S(k)$$

(ii) Neyman-Barton

$$B_p = k[\overline{V} - \tfrac{1}{2}, \overline{V^2} - \tfrac{1}{3}, \ldots, \overline{V^p} - \tfrac{1}{p+1}] \tilde{\Sigma}^{-1} \begin{vmatrix} \overline{V} - 1/2 \\ \overline{V^2} - 1/3 \\ \vdots \\ \overline{V^p} - 1/p+1 \end{vmatrix} \underset{PN}{\approx} \chi^2_p$$

where $\overline{V^r} = \frac{1}{k} \sum_{j=1}^{k} \left[\frac{\mu_0(W_j)}{\mu_0(T^*)}\right]^r$

and
$$\tilde{\Sigma} = (\sigma_{rs})$$

$$\sigma_{rs} = \frac{rs}{(r+s+1)(r+1)(s+1)}$$

Detection Rules: (i) (A.I.1.) Decide N + S iff $D_k > d_{k,\alpha}$

(ii) (A.I.2.) Decide N + S iff $B_p > \chi^2_{p}, 1 - \alpha$

For Data Set 1:

T^*	$N(T^*) = k$	D_k	C.V. $(d_{k,.01})$	Decide
3.00	22	0.17	0.337	PN
4.50	47	0.10	0.238	PN

B_2	C.V. $(\chi^2_{2,.99})$	Decide	B_3	C.V. $(\chi^2_{3,.99})$	Decide
0.75	9.21	PN	0.81	11.35	PN
0.42		PN	0.84		PN

For Data Set 3:

T^*	$N(T^*) = k$	D_k	C.V. $(d_{k,.01})$	Decide
4.50	16	0.17	0.392	PN
10.00	21	0.32	0.285	N+S

B_2	C.V. $(\chi^2_{2,.99})$	Decide	B_3	C.V. $(\chi^2_{3,.99})$	Decide
4.59	9.21	PN	5.16	11.35	PN
14.67		N+S	16.96		N+S

Case A.II: Observe $N(t)$ until Nth event. The data is of the form $W_1, W_2, \ldots, W_N$.

Detector Statistics:

(i) Kolmogorov-Smirnov

$$D_{N-1} = \sup_{0<z<1} \left| \frac{1}{N-1} \sum_{j=1}^{N-1} \varepsilon \left[z - \frac{\mu_0(W_j)}{\mu_0(W_N)}\right] - z \right| \underset{PN}{\sim} \text{K-S}(N-1)$$

(ii) Neyman-Barton

$$B_p = (N-1)\left[\overline{V} - 1/2,\ \overline{V^2} - 1/3,\ \ldots,\ \overline{V^p} - 1/p+1\right] \tilde{\Sigma}^{-1} \begin{vmatrix} \overline{V} - 1/2 \\ \overline{V^2} - 1/3 \\ \vdots \\ \overline{V^p} - 1/p+1 \end{vmatrix} \underset{PN}{\approx} \chi_p^2$$

where $\overline{V^r} = \frac{1}{N-1} \sum_{j=1}^{N-1} \left[\frac{\mu_0(W_j)}{\mu_0(W_N)}\right]^r$ and $\tilde{\Sigma} = (\sigma_{rs})$

$$\sigma_{rs} = \frac{rs}{(r+s+1)(r+1)(s+1)}$$

Detection Rules:

(i) (A.II.1.) Decide $N + S$ iff $D_{N-1} > d_{N-1,\alpha}$

(ii) (A.II.2.) Decide $N + S$ iff $B_p > \chi^2_{p,1-\alpha}$

For Data Set 1 ($N = 50$) the following values are obtained:

(i) $D_{49} = 0.06$ Decide PN, since critical value is $d_{49,.01} = 0.23$

(ii) $B_2 = 0.60$ Decide PN, since critical value is $\chi^2_{2,.99} = 0.21$

$B_3 = 1.16$ Decide PN, since critical value is $\chi^2_{3,.99} = 11.35$.

For Data Set 3 (N = 50) the following values are obtained (same critical values as above):

(i) $D_{49} = 0.04$ Decide PN.

(ii) $B_2 = 28.49$ Decide N + S.

$B_3 = 29.86$ Decide N + S.

Case A.III: Regular Sampling

Observe $N(t)$ at regular intervals of time $\Delta, 2\Delta, \ldots, k\Delta$

Data Set 1 ($\Delta = 0.50$)	Data Set 2 ($\Delta = 1.00$)
$N(\Delta) = 0$	$N(\Delta) = 0$
$N(2\Delta) = 2$	$N(2\Delta) = 2$
$N(3\Delta) = 6$	$N(3\Delta) = 7$
$N(4\Delta) = 10$	$N(4\Delta) = 14$
$N(5\Delta) = 15$	$N(5\Delta) = 21$
$N(6\Delta) = 22$	$N(6\Delta) = 34$
$N(7\Delta) = 29$	$N(7\Delta) = 48$
$N(8\Delta) = 36$	
$N(9\Delta) = 47$	

Detector Statistic: $$T = \sum_{j=1}^{k} \frac{\left(y_j - \frac{b_j N(k\Delta)}{\mu_0(k\Delta)}\right)^2}{\frac{b_j N(k\Delta)}{\mu_0(k\Delta)}} \underset{PN}{\approx} \chi^2_{k-1}$$

where $y_j = N(j\Delta) - N([j-1]\Delta) \quad j = 1, \ldots, k$

and $b_j = \mu_0(j\Delta) - \mu_0([j-1]\Delta)$

<u>Detection Rule</u>: Decide $N + S$ iff $T > \chi^2_{k-1,1-\alpha}$

For Data Set 1 ($k = 9$, $\Delta = 0.50$) we compute $T = 1.35$. Since the critical value is $\chi^2_{8,.99} = 20.1$ one decides PN.

<u>Case A.IV</u>: Equal-Distance Sampling

Observe $N(t)$ at times $t_1, t_2, \ldots, t_k$ where the t's are chosen so that $\mu_0(t_j) - \mu_0(t_{j-1})$ are equal for every $j = 1, \ldots, k$; $t_0 = 0$.

If we assume $\mu_0(t) = t^2$, a convenient choice for the observation times are $t_j = \sqrt{j}$ for $j = 1, \ldots, k$.

$[\mu_0(t_j) - \mu_0(t_{j-1}) = 1$ for all $j]$.

Data Set 1 (k = 10)	Data Set 2 (k = 15)	Data Set 4 (k = 15)
$N(t_1) = 2$	$N(t_1) = 9$	$N(t_1) = 0$
$N(t_2) = 5$	$N(t_2) = 0$	$N(t_2) = 0$
$N(t_3) = 9$	$N(t_3) = 1$	$N(t_3) = 1$
$N(t_4) = 10$	$N(t_4) = 2$	$N(t_4) = 1$
$N(t_5) = 12$	$N(t_5) = 2$	$N(t_5) = 1$
$N(t_6) = 15$	$N(t_6) = 4$	$N(t_6) = 1$
$N(t_7) = 20$	$N(t_7) = 4$	$N(t_7) = 1$
$N(t_8) = 21$	$N(t_8) = 5$	$N(t_8) = 1$
$N(t_9) = 22$	$N(t_9) = 7$	$N(t_9) = 1$
$N(t_{10}) = 24$	$N(t_{10}) = 7$	$N(t_{10}) = 1$
	$N(t_{11}) = 8$	$N(t_{11}) = 3$
	$N(t_{12}) = 9$	$N(t_{12}) = 3$
	$N(t_{13}) = 9$	$N(t_{13}) = 3$
	$N(t_{14}) = 11$	$N(t_{14}) = 4$
	$N(t_{15}) = 12$	$N(t_{15}) = 6$

Detector Statistic: $T = \frac{1}{\overline{Y}} \sum_{j=1}^{k} (Y_j - \overline{Y})^2 \underset{PN}{\sim} \chi^2_{k-1}$ where

$Y_j = N(t_j) - N(t_{j-1})$.

Detection Rule: Decide $N + S$ iff $T > \chi^2_{k-1,\, 1-\alpha}$

For Data Set 1, $\overline{Y} = 2.4$ and we get $T = 6.77$. Since the critical value is $\chi^2_{9,.99} = 21.67$, one decides PN. For Data Set 2, $\overline{Y} = 0.80$ and we get $T = 10.45$. Since the critical value is $\chi^2_{14,.99} = 29.14$, one decides PN.

Case A.V.: Same-Shape Sampling.

Observe NHPP $N(t)$ with mean function $\beta\mu_0(t)$ at times $w_1^*, w_2^*, \ldots$ w_k^*, where the w^*'s are the waiting times of an independent NHPP $N^*(t)$ with mean function $\mu_0(t)$. For these examples, Data Set 1 is observed at the waiting times of Data Set 2 $(\mu(t) = t^2)$. Also, two more independent processes with $\mu(t) = t^2$ were generated to provide times of observation for Data Sets 4 and 5.

Data Set 1	Data Set 4	Data Set 5
$N(w_1^*) = 6$	$N(w_1^*) = 0$	$N(w_1^*) = 0$
$N(w_2^*) = 9$	$N(w_2^*) = 1$	$N(w_2^*) = 0$
$N(w_3^*) = 12$	$N(w_3^*) = 1$	$N(w_3^*) = 0$
$N(w_4^*) = 15$	$N(w_4^*) = 1$	$N(w_4^*) = 1$
$N(w_5^*) = 20$	$N(w_5^*) = 1$	$N(w_5^*) = 1$
$N(w_6^*) = 21$	$N(w_6^*) = 1$	$N(w_6^*) = 1$
$N(w_7^*) = 21$	$N(w_7^*) = 1$	$N(w_7^*) = 1$
$N(w_8^*) = 28$	$N(w_8^*) = 1$	$N(w_8^*) = 1$
$N(w_9^*) = 27$	$N(w_9^*) = 1$	$N(w_9^*) = 1$
$N(w_{10}^*) = 32$	$N(w_{10}^*) = 3$	$N(w_{10}^*) = 1$
$N(w_{11}^*) = 32$	$N(w_{11}^*) = 3$	$N(w_{11}^*) = 1$
$N(w_{12}^*) = 33$	$N(w_{12}^*) = 3$	$N(w_{12}^*) = 1$
$N(w_{13}^*) = 34$	$N(w_{13}^*) = 3$	$N(w_{13}^*) = 1$
$N(w_{14}^*) = 34$	$N(w_{14}^*) = 4$	$N(w_{14}^*) = 2$
$N(w_{15}^*) = 44$	$N(w_{15}^*) = 4$	$N(w_{15}^*) = 2$
$N(w_{16}^*) = 47$	$N(w_{16}^*) = 5$	$N(w_{16}^*) = 2$

<u>Detector Statistic</u>: $T = \sum_{i=1}^{s} \frac{(n_i - k\hat{p}(c_i)}{k\hat{p}(c_i)} \underset{PN}{\approx} \chi^2_{s-1}$

where n_i = number of y_i's in C_i (one of s mutually exclusive class covering the non-negative integers).

<u>Detection Rule</u>: Decide N + S iff T $\chi^2_{s-1,\ 1-\alpha}$.

If we take $s = 5$, then for $i = 1, \ldots, 5$ $C_i = \{5j + i - 1:\ j = 0, 1, 2, \ldots\}$. Thus, $C_1 = \{0, 5, 10, 15, \ldots\}$, $C_2 = \{1, 6, 11, 16, \ldots\}$ etc. For Data Set 1 we have $k = 16$ and $N(w^*_{16}) = 47$. The MLE of β is $\hat{\beta} = \frac{N(w^*_{16})}{16} = \frac{47}{16} = 2.94$. We also have $\hat{p} = (1 + \hat{\beta})^{-1} = 0.254$ and $\hat{p}(C_i) = \hat{p}(\hat{q})^{i-1}\ (1-\hat{q}^N)^{-1}$ for $i = 1, \ldots, 5$. The $\hat{p}(C_i)$ and the $k\hat{p}(C_i)$ (with $k = 16$) are computed as:

$\hat{p}(C_1) = 0.330$	$k\hat{p}(C_1) = 5.280$
$\hat{p}(C_2) = 0.246$	$k\hat{p}(C_2) = 3.936$
$\hat{p}(C_3) = 0.184$	$k\hat{p}(C_3) = 2.944$
$\hat{p}(C_4) = 0.137$	$k\hat{p}(C_4) = 2.192$
$\hat{p}(C_5) = 0.102$	$k\hat{p}(C_5) = 1.632$

The n_i are:

$n_1 = 7$

$n_2 = 5$

$n_3 = 0$

$n_4 = 4$

$n_5 = 0$

The computed statistic value is $T = 6.915$ and the critical value is $\chi^2_{4,.99} = 13.277$. Therefore, decide PN.

Problem B: Goodness-of-fit.

$$\text{PN:} \quad \mu(\cdot) = \beta_0\mu_0(\cdot)_{(\beta=\beta_0)} \quad \text{vs.} \quad \text{N + S:} \quad \mu(\cdot) \neq \beta_0\mu_0(\cdot)_{(\beta\neq\beta_0)}$$

Case B.I.: Observe $N(t)$ until time T^*.

Detector Statistic: $N(T^*) \underset{PN}{\sim} P_0(\beta_0\mu_0(T^*))$

Detection Rule: Decide N + S iff $N(T^*) \leq a_1$ or $N(T^*) \geq a_2$ where a_1 and a_2 are appropriate percentiles of the Poisson distribution.

For Data Set 1, consider the following detection problem:

$$\text{PN:} \quad \beta = 1, \quad \text{vs.} \quad \text{N + S:} \quad \beta \neq 1$$

If $T^* = 3.00$, then we observe $N(3.00) = 22$. Also, $\mu(T^*) = \beta_0\mu_0(T^*) = (1)(3.00)^2 = 9.00$. From Poisson tables, the critical values are $a_1 = 2$ and $a_2 = 18$ (with FAR $\alpha = .0115$). Thus, one decides N + S.

Case B.II.: Observe $N(t)$ until Nth event.

Detector Statistic: $T = 2\beta_0\mu_0(W_N) \underset{PN}{\sim} \chi^2_{2N}$.

Detection Rule: Decide N + S iff $T < b_1$ or $T > b_2$ where b_1 and b_2 are appropriate percentiles of the χ^2_{2N}-distribution.

For Data Set 2, assume $\mu_0(t) = t^2$.

We observe $W_{50} = 7.1849$ and consider the following three detection problems:

(i) PN: $\beta = 2$ vs. N + S: $\beta \neq 2$

(ii) PN: $\beta = 1$ vs. N + S: $\beta \neq 1$

(iii) PN: $\beta = 0.5$ vs. N + S: $\beta \neq 0.5$

In each case the critical values are

$$b_1 = \chi^2_{100,.005} = 67.328 \quad \text{and}$$

$$b_2 = \chi^2_{100,.995} = 140.169.$$

We then compute

(i) $T = 2(2)(7.1844)^2 = 206.49$ and decide N + S.

(ii) $T = 2(1)(7.1849)^2 = 103.25$ and decide PN.

(iii) $T = 2(.05)(7.1849)^2 = 51.62$ and decide N + S.

<u>Case B.III.</u>: Regular Sampling (See data on page 7).

<u>Detector Statistic</u>: $$T = \sum_{j=1}^{k} \frac{(y_j - b_j\beta_0)^2}{b_j\beta_0} \underset{PN}{\approx} \chi^2_k$$

<u>Detection Rule</u>: Decide N + S iff $T > \chi^2_{k,1-\alpha}$

For Data Set 1, we will consider:

PN: $\beta = 2.5$ vs. N + S: $\beta \neq 2.5$

j	y_j	$b_j\beta_0$	$(y_j-b_j\beta_0)^2$	$\frac{(y_j-b_j\beta_0)^2}{b_j\beta_0}$
1	0	0.625	0.391	0.625
2	2	1.875	0.125	0.067
3	4	3.125	0.766	0.245
4	4	4.375	0.141	0.032
5	5	5.625	0.391	0.069
6	7	6.875	0.016	0.002
7	7	8.125	1.266	0.156
8	7	9.375	5.641	0.602
9	11	10.625	0.141	0.013
			T =	1.811

Since the critical value is $\chi^2_{9,.01} = 2.088$ one decides PN.

<u>Case B.IV.</u>: Equal-Distance Sampling (see data on page 9).

<u>Detector Statistic</u>: $N(t_k) \underset{PN}{\sim} P_0(\beta_0\mu_0(t_k))$

<u>Detection Rule</u>: Decide $N + S$ iff $N(t_k) \leq a_1$ or $N(t_k) \geq a_2$ where a_1 and a_2 are appropriate percentiles of the Poisson distribution.

Suppose one is interested in testing

$$PN: \beta = 1 \quad \text{vs.} \quad N + S: \beta \neq 1$$

for Data Sets 1 and 2. Taking $k = 10$, one has $N_1(t_{10}) = 24$

and $N_2(t_{10}) = 7$. Since $t_j = \sqrt{j}$, it follows that $\beta_0 \mu_0(t_k) = (1)(\sqrt{10})^2 = 10$. From Poisson tables, the critical values are $a_1 = 2$ and $a_2 = 20$ (with FAR $\alpha = .0063$), so one decides N + S for Data Set 1 and PN for Data Set 2.

Case B.V.: Same-Shape Sampling (see data on page 10)

Detector Statistic: $N(w_k^*) \underset{PN}{\sim} NB(k,p)$ where $p = [1 + \beta_0]^{-1}$.

Detection Rule: Decide N + S iff $N(w_k^*) \leq b_1$ or $N(w_k^*) \geq b_2$, are appropriate percentiles of the negative Binomial distribution.

Consider the following problem:

$$\text{PN:} \quad \beta = 1 \quad \text{vs.} \quad \text{N + S:} \quad \beta \neq 1$$

For Data Set 1, we observe $N(w_{16}^*) = 47$. Thus $k = 16$ and $p = \frac{1}{1 + \beta_0} = \frac{1}{2}$.

To compute critical values, we must find b_1 and b_2 such that $P(x \leq b_1) = \frac{\alpha}{2}$ and $P(x \geq b_2) = \frac{\alpha}{2}$ where $X \sim NB(16, \frac{1}{2})$. A corrected (Gram-Charlier) Poisson approximation will be used (see Johnson and Kotz, Discrete Distributions, pg. 129). $P(X \leq b_1) \overset{\sim}{\sim} P(Y \leq b_1) - \frac{b_1 - NP}{2(1 + P)} P(Y = b_1)$

where $N = 16$, $P = \frac{p}{1 - p} = 1$ and $Y \sim P_0(NP)$. Thus,

$$P(X \leq b_1) \overset{\sim}{\sim} P(Y \leq b_1) - \frac{b_1 - 16}{4} P(Y = b_1)$$

where $Y \sim P_0(16)$.

From Poisson tables,

$$P(X \leq 6) \overset{\sim}{\sim} .0040 - \frac{6 - 16}{4}(.0026) = .0105$$

so $b_1 = 6$ with approximate tail probability .0105.

To find b_2, we must satisfy:

$$P(X \geq b_2) = 1 - P(X \leq b_2 - 1) = \frac{\alpha}{2}$$

$$\text{or} \quad P(X \leq b_2 - 1) = 1 - \frac{\alpha}{2}$$

$$P(X \leq b_2 - 1) \overset{\sim}{\sim} P(Y \leq b_2 - 1) - \frac{(b_2 - 1) - NP}{2(1 + P)} P(Y = b_2 - 1)$$

where N, P and Y are defined as above. Thus,

$$P(X \leq 28) \overset{\sim}{\sim} .9978 - 3(.0019) = .9921$$

so $b_2 = 29$ with approximate tail probability .0079.

The critical values are therefore $b_1 = 6$ and $b_2 = 29$ with FAR $\alpha = .0184$. Since we observed $N(w^*_{16}) = 47$, decide N + S.

<u>Problem C</u>: Two-sample

$$\text{PN:} \quad L_1 = L_{2(\beta_1=\beta_2)} \quad \text{vs. N + S:} \quad L_1 \neq L_{2(\beta_1 \neq \beta_2)}$$

<u>Case C.I.</u>: Observe $N(t)$ until time T^*.

<u>Detection Statistic</u>: Conditionally given $N(T^*) + M(T^*) = N$,

$$N(T^*) \sim B(N, 1/2) \quad \text{(under PN)}$$

<u>Detection Rule</u>: Decide N + S iff $N(T^*) < c_1$ or $N(T^*) > c_2$,

can be determined using the Normal approximation to the Binomial.

$$c_1 = N(1/2) - z_{\alpha/2} \sqrt{N(1/2)(1/2)}$$

$$c_2 = N(1/2) + z_{\alpha/2} \sqrt{N(1/2)(1/2)}$$

If Data Sets 1 and 2 are observed until $T^* = 3.00$, we arrive at $N(T^*) = 22$ and $M(T^*) = 7$. For $\alpha = .01$

$$c_1 = 29(1/2) - 2.576 \sqrt{29(1/2)(1/2)} = 7.56$$

$$c_2 = 29(1/2) + 2.576 \sqrt{29(1/2)(1/2)} = 21.44$$

Since $N(T^*) = 22$, one decides $N + S$.

<u>Case C.II.</u>: Observe $N(t)$ until Nth event.

<u>Detector Statistic</u>: $$T = \frac{m\mu_0(W_n)}{n\mu_0(W_m^*)} \underset{PN}{\sim} F(2n, 2m)$$

<u>Detection Rule</u>: Decide $N + S$ iff $T < F_{(2n, 2m, \alpha/2)}$ or $T > F_{(2n, 2m, 1-\alpha/2)}$.

For Data Sets 1 and 2 assume

$$L_1, L_2 \in \Omega(\mu_0) \quad \text{and} \quad \mu_0(t) = t^2.$$

Taking $n = 40$ and $m = 50$ we find:

$$W_{40} = 4.1229 \quad \text{and} \quad W_{50}^* = 7.1849.$$

Thus, $T = \dfrac{50(4.1229)^2}{40(7.1849)^2} = 0.412.$

Since the critical values are $F_{80,100,.005} = 0.70$ and $F_{80,100,.995} = 1.41$ decide $N + S$.

<u>Case C.III.</u>: Regular Sampling

<u>Detector Statistic</u>: Conditionally given

$$N(n\Delta_1) + M(m\Delta_2) = N, \quad N(n\Delta_1) \sim B(N,p) \quad \text{(under PN)}$$

where $p = \dfrac{\mu_0(n\Delta_1)}{\mu_0(n\Delta_1) + \mu_0(m\Delta_2)}$.

<u>Detection Rule</u>: Decide $N + S$ iff $N(n\Delta_1) < c_1$ or $N(n\Delta_1) > c_2$, where c_1 and c_2 can be determined using the normal approximation to the Binomial:

$$c_1 = Np - z_{\alpha/2} \sqrt{Np(1 - p)}$$

$$c_2 = Np + z_{\alpha/2} \sqrt{Np(1 - p)}$$

If Data Sets 1 and 2 are observed (with $n = 9$, $\Delta_1 = 0.50$, $m = 7$, $\Delta_2 = 1.00$) we arrive at $N(n\Delta_1) = 47$ and $M(m\Delta_2) = 48$. Thus, $N = N(n\Delta_1) + M(m\Delta_2) = 47 + 48 = 95$ and

$$= \frac{\mu_0(n\Delta_1)}{\mu_0(n\Delta_1) + \mu_0(m\Delta_2)} = \frac{(4.5)^2}{(4.5)^2 + (7.0)^2} = 0.29$$

$$c_1 = (95)(0.29) - (2.576)\ \sqrt{(95)(0.29)(0.71)} = 16.36$$

$$c_2 = (95)(0.29) + (2.576)\ \sqrt{95(0.29)(0.71)} = 39.20$$

Since $N(n\Delta_1) = 47$, one decides $N + S$.

<u>Case C.IV.</u>: Equal-Distance Sampling

<u>Detector Statistic</u>: Conditionally given

$$N(t_n) + M(t_m) = N, \quad N(t_n) \sim B(N,p)$$

where

$$p = \frac{\mu_0(t_n)}{\mu_0(t_n) + \mu_0(t_m)}$$

<u>Detection Rule</u>: Decide $N + S$ iff $N(t_n) < b_1$ or $N(t_n) > b_2$ where b_1 and b_2 can be determined by the Normal approximation to the Binomial.

$$b_1 = Np - z_{\alpha/2}\ \sqrt{Np(1 - p)}$$

$$b_2 = Np + z_{\alpha/2}\ \sqrt{Np(1 - p)}$$

For Data Sets 1 and 2, with $n = 10$ and $m = 15$, we observe $N(t_{10}) = 24$ and $M(t_{15}) = 12$. Since $\mu_0(t) = t^2$ and $t_j = \sqrt{j}$, we have

$$p = \frac{\mu_0(t_{10})}{\mu_0(t_{10}) + \mu_0(t_{15})} = \frac{(\sqrt{10})^2}{(\sqrt{10})^2 + (\sqrt{15})^2} = \frac{10}{25} = 0.4$$

Thus (for $\alpha = .01$),

$$b_1 = (36)(.4) - 2.576\sqrt{36(.4)(.6)} = 6.83$$

$$b_2 = (36)(.4) + 2.576\sqrt{35(.4)(.6)} = 21.97$$

Since $N(t_{10}) = 24$, one decides $N + S$.

<u>Case C.V.</u>: Same-Shape Sampling

<u>Detector Statistic</u>: Conditionally given

$$N(W_n^*) + M(V_m^*) = k, \qquad N(W_n^*) \underset{PN}{\sim} F^*$$

where

$$P_{F^*}\{N(W_n^*) = s \mid N(W_n^*) + M(V_m^*) = k\} = \frac{\binom{n+s-1}{s}\binom{m+k-s-1}{k-s}}{\binom{n+m+k-1}{k}}$$

<u>Detection Rule</u>: Decide $N + S$ iff $N(W_n^*) < c_1^*$ or $N(W_n^*) > c_2^*$ where c_1^* and c_2^* are the largest and smallest non-negative integers such that

$$\sum_{s=0}^{c_1^*-1} P_{F^*}[N(W_n^*) = s \mid N(W_n^*) + M(V_m^*) = k] \leq \frac{\alpha}{2}$$

$$\sum_{s=c_2^*+1}^{k} P_{F^*}[N(W_n^*) = s \mid N(W_n^*) + M(V_m^*) = k] \leq \frac{\alpha}{2}$$

For Data Sets 4 and 5 we observe $N(W_{16}^*) = 5$ and $M(V_{16}^*) = 2$, so $k = N(W_{16}^*) + M(V_{16}^*) = 7$. To compute critical values c_1^* and c_2^*:

$$\frac{\binom{15}{0}\binom{22}{7}}{\binom{38}{7}} = \frac{170544}{12620256} = .0135$$

Thus, for FAR $\alpha = .027$, the critical values are $c_1^* = 1$ and $c_2^* = 6$. Since $N(W_{16}^*) = 5$, decide PN.

Problem D: Constant Intensity Rate.

$$\text{PN:} \quad \beta_1 = \gamma_0\beta_2 \qquad \text{vs.} \qquad N + S\text{:} \quad \beta_1 \neq \gamma_0\beta_2$$

Case D.I.: Observe $N(t)$ until time T^*.

Detector Statistic: Conditionally given

$$N(T^*) + M(T^*) = N, \quad N(T^*) \sim B(N,p) \qquad \text{(under PN)}$$

where $p = \frac{\gamma_0}{1 + \gamma_0}$

Detection Rule: Decide $N + S$ iff $N(T^*) < c_1$ or $M(T^*) > c_2$, where c_1 and c_2 are determined as in Case C.I, replacing $p = 1/2$ with $p = \frac{\gamma_0}{1 + \gamma_0}$. Suppose we take $\gamma_0 = 2.5$, then $p = \frac{2.5}{1 + 2.5} = 0.71$.

Using Data Sets 1 and 2, consider the following problem:

$$\text{PN:} \quad \mu_1(t) = (2.5)\mu_2(t) \quad \text{vs.} \quad N + S\text{:} \quad \mu_1(t) \neq (2.5)\mu_2(t)$$

As before, we observe $N(T^*) = 22$ and $M(T^*) = 7$ for $T^* = 3.00$.

The critical values are computed below: $(\alpha = .01)$.

$$c_1 = 29(0.71) - 2.576 \sqrt{29(0.71)(0.29)} = 14.45$$

$$c_2 = 29(0.71) + 2.576 \sqrt{29(0.71)(0.29)} = 26.98$$

Since $N(T^*) = 22$, one decides PN.

<u>Case D.II.</u>: Observe $N(t)$ until Nth event.

<u>Detector Statistic</u>: $T = \dfrac{m\gamma_0\mu_0(W_n)}{n\mu_0(W_n^*)} \underset{PN}{\sim} F(2n, 2m)$

<u>Detection Rule</u>: Decide $N + S$ iff $T < F_{(2n, 2m, \alpha/2)}$

or $T > F_{(2n, 2m, 1-\alpha/2)}$

As in Case C.II, we observe $W_{40} = 4.1229$ and $W_{50}^* = 7.1849$. If $\gamma_0 = 2.5$, the computed value of the detector statistic is

$$T = \frac{50(2.5)\ (4.1229)^2}{40(7.1849)^2} = 1.029$$

Since the critical values are $F_{80,100,.005} = 0.70$ and $F_{80,100,.995} = 1.41$ decide PN.

<u>Case D.III.</u>: Regular Sampling

<u>Detector Statistic</u>: Conditionally given

$$N(n\Delta_1) + M(m\Delta_2) = N, \quad N(n\Delta_1) \sim B(N,p) \quad \text{(under PN)}$$

where $$p = \frac{\gamma_0 \mu_0(n\Delta_1)}{\gamma_0 \mu_0(n\Delta_1) + \mu_0(m\Delta_2)}$$

Detection Rule: Decide N + S iff $N(n\Delta_1) < c_1$ or $N(n\Delta_1) > c_2$, where c_1 and c_2 are computed as in Case C.III.

For Data Sets 1 and 2 we consider:

$$\text{PN:} \quad \beta_1 = (2.5)\beta_2 \qquad \text{vs.} \qquad \text{N + S:} \quad \beta_1 \neq (2.5)\beta_2$$

Again, $N = 95$ and $p = \dfrac{(2.5)(4.5)^2}{(2.5)(4.5)^2 + (7.0)^2} = 0.51$.

Thus, (for $\alpha = .01$)

$$c_1 = (95)(.51) - (2.576)\sqrt{95(.51)(.49)} = 35.75$$

$$c_2 = (95)(.51) + (2.576)\sqrt{95(.51)(.49)} = 60.80$$

Since $N(n\Delta_1) = 47$, one decides PN.

Case D.IV.: Equal-Distance Sampling

Detector Statistics: Conditionally given

$$N(t_n) + M(t_m) = N, \qquad N(t_n) \underset{\text{PN}}{\sim} B(N,p)$$

where

$$p = \frac{\gamma_0 \mu_0(t_n)}{\gamma_0 \mu_0(t_n) + \mu_0(t_m)}$$

<u>Detection Rule</u>: Decide N + S iff $N(t_n) < b_1$ or $N(t_n) > b_2$ where b_1 and b_2 are determined as in Case C.IV. Suppose one wants to test

$$\text{PN:} \quad \beta_1 = 2.5\beta_2 \qquad \text{vs.} \quad \text{N + S:} \quad \beta_1 \neq 2.5\beta_2$$

for Data Sets 1 and 2. For $n = 10$ and $m = 15$, we observe $N(t_{10}) = 24$ and $M(t_{15}) = 12$. Since $\mu_0(t) = t^2$ and $t_j = \sqrt{j}$, we have

$$p = \frac{\gamma_0\mu_0(t_{10})}{\gamma_0\mu_0(t_{10})+\mu_0(t_{15})} = \frac{(2.5)(\sqrt{10})^2}{(2.5)(\sqrt{10})^2+(\sqrt{15})^2} = \frac{(2.5)(10)}{(2.5)(10)+15} = 0.63$$

Thus (for $\alpha = .01$),

$$b_1 = 36(.63) - 2.576 \sqrt{36(.63)(.38)} \doteq 15.02$$

$$b_2 = 36(.63) + 2.576 \sqrt{36(.63)(.38)} = 29.98$$

Since $N(t_{10}) = 24$, one decides PN.

<u>Case D.V.</u>: Same-Shape Sampling

Here, two methods are available: Method 1, based on the likelihood ratio and Method 2, using a normal approximation to the Binomial distribution. Sampling Plan V is slightly modified for Method 2. Instead of generating two independent "noise" processes to provide times of observations, we observe the two processes of interest at the waiting time of a single independent same-shape process.

Method 1 (LH ratio)

Detector Statistic: $L = -2 \log \frac{L_0}{L_1} \underset{PN}{\sim} X_1^2$

where

$$L_1 = \left(\frac{1}{1+\bar{y}_{1.}}\right)^k \left(\frac{1}{1+\bar{y}_{2.}}\right)^k \left(\frac{\bar{y}_1}{1+\bar{y}_{1.}}\right)^{N(W_k^*)} \left(\frac{\bar{y}_{2.}}{1+\bar{y}_{2.}}\right)^{M(V_k^*)}$$

where

$$\bar{y}_{1.} = \frac{N(W_k^*)}{k}$$

$$\bar{y}_{2.} = \frac{M(V_k^*)}{k}$$

$$L_0 = \gamma_0^{N(W_k^*)} \left[\frac{\hat{\hat{p}}^{2k}(1-\hat{\hat{p}})^{N(W_k^*)+M(V_k^*)}}{[\gamma_0 + (1-\gamma_0)\hat{\hat{p}}]^{k+n(w_k^*)}} \right]$$

and

$$\hat{\hat{p}} = \frac{(3\gamma_0+\bar{y}_{2.}\gamma_0+\bar{y}_{1.}-1) \pm \sqrt{(3\gamma_0+\bar{y}_{2.}\gamma_0+\bar{y}_{1.}-1)^2 + 8\gamma_0[1-\gamma_0+\bar{y}_{2.}(1-\gamma_0)]}}{2[\gamma_0 - 1 - y_{2.}(1-\gamma_0)]}$$

An obvious requirement is for $0 \leq \hat{\hat{p}} \leq 1$, and in many cases (i.e., only 1 root on [0,1]) the decision whether to use "+" or "-" in the above formula can be based on this. In the case of two roots on [0,1], further investigation is needed.

<u>Detection Rule</u>: Decide N + S iff $L > \chi^2_{1,1-\alpha}$. Consider the following detection problem.

$$\text{PN:}\quad \beta_1 = 5\beta_4 \qquad \text{vs.} \qquad \text{N + S:}\quad \beta_1 \neq 5\beta_4$$

where β_1 and β_4 are associated with Data Sets 1 and 4, respectively. If we observe $N(W^*_{16}) = 47$, $M(V^*_{16}) = 5$, then

$$\bar{y}_{1.} = \frac{47}{16} = 2.94$$

and

$$\bar{y}_{4.} = \frac{5}{16} = 0.31$$

We now compute:

$$\hat{\hat{p}} = 0.6665$$

$$L_1 = 3.0919 \times 10^{-21}$$

$$L_0 = 1.6743 \times 10^{-21}$$

$L = -2 \log \frac{L_0}{L_1} = -2 \log (1.8467) = 1.2268$. Since the critical value is $\chi^2_{1,.95} = 3.84$, deciee PN.

<u>Method 2</u>. (Binomial type)

<u>Detector Statistic</u>: $$Z = \frac{N(w^*_k) - \gamma_0 M(w^*_k)}{\sqrt{[N(w^*_k) + M(w^*_k)]\gamma_0}} \underset{\text{PN}}{\approx} \Phi$$

The above statistic is derived using the fact that, given $N(w^*_k) + M(w^*_k) = N$,

$$N(w_k^*) \underset{PN}{\overset{c.}{\sim}} B(N, \frac{\gamma_0}{1 + \gamma_0}) \quad \text{(for large } N)$$

Detection Rule: Decide N + S iff $Z > z_{\alpha/2}$ or $Z < -z_{\alpha/2}$. Again, consider the problem.

$$PN: \quad \beta_1 = 5\beta_4 \qquad \text{vs.} \qquad N + S: \quad \beta_1 \neq 5\beta_4$$

This time we observe both Data Sets 1 and 4 at the waiting times from Data Set 2. If we take $k = 16$, $w_{16}^* = 4.4830$ and our final observations are

$$N(w_{16}^*) = 47 \quad \text{and} \quad M(w_{16}^*) = 7.$$

Thus,

$$Z = \frac{47 - 5(7)}{\sqrt{(47 + 7)5}} = 0.731$$

and we decide PN.

Consider two epileptics (patients 1 and 2) who have seizure times given on pages 2-3. Assume the processes can be modeled as NHPP's with mean function of the form

$$\mu_i(t) = \beta_i t^{0.5} \qquad i = 1,2$$

We want to test the following situation:

$$PN: \quad \beta_1 = 0.25\beta_2 \qquad \text{vs.} \qquad N + S: \quad \beta_1 \neq 0.25\beta_2$$

An independent NHPP with mean function $\mu(t) = t^{0.5}$ is generated to provide times of observation. For $k = 50$, we have

$$w_{50}^{*} = 239.40 \text{ min.} = 4.0 \text{ hours}$$

and we observe

$$N(w_{50}^{*}) = 11 \quad \text{and} \quad M(w_{50}^{*}) = 14$$

Thus,

$$Z = \frac{11 - (0.25)14}{\sqrt{(11 + 14)(0.25)}} = 3.0$$

and we decide $N + S$.

<u>Problem E</u>: c-sample

$$\text{PN:} \quad L_1 = \ldots = L_c \quad \text{vs.} \quad N + S: \quad L_i\text{'s not all equal}$$

<u>Case E.I.</u>: Observe $N(t)$ until time T^*.

<u>Detector Statistic</u>: $$T = \sum_{j=1}^{c} \frac{(N_j(T^*) - \overline{N})^2}{\overline{N}} \underset{PN}{\sim} \chi^2_{c-1}$$

where $\overline{N} = \frac{1}{c} \sum_{j=1}^{c} N_j(T^*)$.

<u>Detection Rule</u>: Decide $N + S$ iff $T > \chi^2_{c-1,1-\alpha}$.

Consider the case $c = 4$, where the four samples consist of the points of Data Set 3 observed at time 0 - 4.0, 4.0 - 8.0, 8.0 - 12.0, 12.0 - 16.0. Thus $T^* = 4.0$ and we observe $N_1(T^*) = 11$, $N_2(T^*) = 17$, $N_3(T^*) = 12$ and $N_4(T^*) = 10$. ($\overline{N} = 12.5$). The computed statistic value is $T = 2.32$. Since the critical value is $\chi^2_{3,.99} = 11.35$, one decides PN.

<u>Case E.II.</u>: Observe $N(t)$ until Nth event.

<u>Detector Statistic</u>: $$T = \frac{\max\{\mu_0(W_{iN}):i = 1,\ldots c\}}{\min\{\mu_0(W_{iN}):i = 1,\ldots c\}}$$

Detection Rule: Decide N + S iff $T > b^*$ where b^* is the $100(1 - \alpha)\%$ point of the $\frac{S^2 \max}{S^2 \min}$ distribution (Hartley).

If Data Set 3 is divided into 4 samples of 12 observations each, we have

$$w_{1,12} = 4.1066$$

$$w_{2,12} = 3.2126$$

$$w_{3,12} = 3.6641$$

$$w_{4,12} = 4.4933$$

Since $L_i \in \Omega(\mu_0)$, $i = 1, \ldots, 4$ with $\mu_0(t) = t$ we compute

$$T = \frac{4.4933}{3.2126} = 1.3986$$

From tables of the $\frac{S^2 \max}{S^2 \min}$ distribution, a critical value of $b^* = 7.6$ is obtained (for $\alpha = .01$ and 11 degrees of freedom).

We therefore decide PN.

<u>Case E.III.</u>: Regular Sampling

<u>Detector Statistic</u>: $$T^* = \sum_{j=1}^{k} T_j \underset{PN}{\sim} \chi^2\ (ck - k)$$

where $T_j = \sum_{i=1}^{c} \frac{(N_i(j\Delta) - \overline{N}_j)^2}{\overline{N}_j}$ and $\overline{N}_j = \frac{1}{c} \sum_{i=1}^{c} N_i(j\Delta)$

<u>Detection Rule</u>: Decide N + S iff $T^* > \chi^2_{ck-k,1-\alpha}$

If, for Data Set 3, we take $\Delta = 0.50$ and $k = 10$, $N_1(j\Delta)$, $N_2(j\Delta)$ and $N_3(j\Delta)$ can be generated consectutively:

$j\Delta$	$N_1(j\Delta)$	$N_2(j\Delta)$	$N_3(j\Delta)$	$\overline{N}_j$	T_j
0.50	2	0	2	1.33	2.00
1.00	3	1	5	3.00	2.67
1.50	4	3	7	4.67	1.86
2.00	5	4	9	6.00	2.33
2.50	6	8	10	8.00	1.00
3.00	8	11	12	10.33	0.84
3.50	10	12	12	11.33	0.24
4.00	11	12	13	12.00	0.17
4.50	16	13	13	14.00	0.43
5.00	17	14	14	15.00	0.40

$T^* = 11.94$

Since the critical value is $\chi^2_{20,.99} = 37.57$ one decides PN.

Case E.IV.: Equal-Distance Sampling

Detector Statistic: $W = -2 \log 1$

$$= -2 \sum_{i=1}^{c} N_i(t_k) \log \left[\frac{\sum_{i=1}^{c} N_i(t_k)}{ck}\right] + 2 \sum_{i=1}^{c} N_i(t_k) \log\left[\frac{N_i(t_k)}{k}\right] \underset{PN}{\sim} \chi^2_{c-1}$$

Detection Rule: Decide $N + S$ iff $W > \chi^2_{c-1,1-\alpha}$

Suppose we take $k = 10$ and $c = 3$ with Data Sets 1, 2 and 4 under consideration. Then we have

$$N_1(t_{10}) = 24, \quad N_2(t_{10}) = 7 \quad \text{and} \quad N_3(t_{10}) = 1$$

The statistic is computed as follows:

$$W = -2[32 \log(\tfrac{32}{30})] + 2[24 \log(\tfrac{24}{10}) + 7 \log(\tfrac{7}{10}) + 1 \log(\tfrac{1}{10})] = 28.29$$

Since the critical value is $\chi^2_{2,.99} = 9.21$, one decides $N + S$.

Case E.V.: Same-Shape Sampling

Detector Statistic: $L = -2 \log 1 \underset{PN}{\sim} \chi^2_{c-1}$

where

$$L = 2ck\ \log(k + \overline{X}) - 2c\overline{X}\ \log(\frac{\overline{X}}{k+\overline{X}}) + 2 \sum_{i=1}^{c} X_i\ \log(\frac{X_i}{k + X_i})$$

$$- 2k \sum_{i=1}^{c} \log(k + X_i)$$

and $X_i = N_i(w_k^*)$.

<u>Detection Rule</u>: Decide $N + S$ iff $L > X^2_{c-1,1-\alpha}$

Suppose we consider Data Sets 1, 4, and 5 with $k = 16$. We then have

$X_1 = N_1(w_{16}^*) = 47$ $\qquad$ $\overline{X} = 18.0$

$X_2 = N_2(w_{16}^*) = 5$

$X_3 = N_3(w_{16}^*) = 2$ $\qquad$ $c\overline{X}\ \log(\frac{\overline{X}}{k + \overline{X}}) = 54\log(0.53) = -34.34$

$L = 2(3)(16)\ \log(34.0) - 2(-34.34)$

$+ 2[47\ \log\ (\frac{47}{63}) + 5\ \log\ (\frac{5}{21}) + 2\ \log\ (\frac{2}{18})] - 2(16)[\log 63 + \log 21 + \log 18]$

$= 338.53 + 68.69 - 50.68 - 322.50 = 34.04.$

Since the critical value is $X^2_{2,.95} = 5.991$ decide $N + S$.

END

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